

## **Annotated Guide to Mathematical Resources on Proportional Reasoning**

### ***Research Articles, Chapters and Books***

#### **Developing concepts of ratio and proportion, Van de Walle**

This chapter provides a detailed overview of ratio and proportion, including identification of “big ideas,” connections to other mathematical topics in the middle and high school curriculum, definitions of key terms, exploration of the nature of proportional reasoning and common student difficulties, and exploration of informal and formal activities that help students develop proportional reasoning. Use of ratio tables to develop student capacity to understand, set up and solve proportional reasoning problems is illustrated, along with discussion of the mathematics of the traditional “cross product algorithm.”

#### **Developing ratio concepts: An Asian perspective, Lo, Watanabe, & Cai**

This article introduces how various Asian textbooks develop ratio concepts. Some key ideas include: helping students distinguish multiplicative from additive comparison; identifying a base quantity and using it for comparison; defining and using *the value of the ratio* (i.e.,  $a/b$  in the ratio  $a:b$ ) in order to emphasize the multiplicative relationship within a ratio; and emphasizing conceptual development of equivalent ratios. For example, measuring the same rectangle in different units, and looking at the ratio of width to length in the different units, develops the concept of equivalent ratios (Figure 2). The article also points out difficulties students have with ratio, including distinguishing the multiplicative relationship in ratio from the similar wording used to describe sports scores (e.g., “they won 6 to 3”) and inconsistent use of units (e.g., 1: 137 to describe 1 meter to 137 centimeters). The article also describes the Asian emphasis on use of simplified integer ratios and bar diagrams to build student understanding of ratio. [Note: We have corrected Figure 4 in this article to show the correct problem representation.]

#### **Learning and teaching ratio and proportion, Cramer, Post, & Currier**

This article provides a succinct description of some of the central ideas from research on proportional reasoning. The article begins by presenting a non-proportional situation and reporting that almost 100% of tested pre-service teachers incorrectly interpreted the problem as proportional and tried to solve it using a standard algorithm for proportions (cross-multiply and divide). The article highlights the importance of understanding the multiplicative relationship among quantities in a proportion and the fact that quantities can be compared between or across “measure spaces.” Characteristics of proportional situations are described, including that graphs of proportional situations are linear and run through the origin, that rate pairs always reduce to the same fraction (the unit rate), and that the constant factor in a  $y=mx$  representation is also the unit rate. The article also describes student solution methods, challenges to student problem solving (e.g., problems with non-integer relationships), and multiple problem types.

### **Making connections: A case for proportionality, Cramer & Post**

This article explores the mathematical characteristics of proportional situations and identifies characteristics of proportional reasoners. The authors emphasize the central importance of understanding the multiplicative relationships within proportions and show how to use data tables to illuminate these relationships. Table, graph, and equation are shown as alternative representations of the same data. The article provides examples of proportional and non-proportional situations and documents the difficulty of distinguishing them, using a study of pre-service teachers.

### **Ratios and Proportionality: AAAS, Atlas of Science Literacy**

Chapter 9 of the *Atlas*, “The Mathematical World,” focuses on ratio and proportionality and describes how proportional reasoning ideas relate to science, mathematics, and technology. One of the more compelling attributes of this resource is the map of student understanding (shown at:

[http://www.project2061.org/publications/atlas/sample/9\\_3\\_RP.pdf](http://www.project2061.org/publications/atlas/sample/9_3_RP.pdf)) that divides proportionality along three strands of benchmarks that address the relationship between parts and wholes, experience with and understanding of numerical descriptions and comparisons, and basic computation skills relevant to ratios and proportions. Important ideas highlighted in the map include distinguishing the meanings of the expression  $a/b$ , characterizing relationships as either additive or multiplicative, fractions as a part of something, similarity of figures, and constant ratio.

### **Teaching fractions and ratios for understanding, Lamon [included separately from toolkit binder]**

This resource book summarizes information on development of ideas related to proportional reasoning and supports the translation of this research into useable ideas for the classroom. The book is designed to help teachers build comfort talking to children about mathematics; questions and activities to use with students are described and student solutions are analyzed. Each of the chapters introduces concepts related to proportional reasoning: reflection questions help teachers explore connections among the book topics. We suggest you flip through the book to identify chapters of particular relevance to your work. The components of powerful reasoning diagram on page 9, and chapters 1, 3, 5, 9, and 15-18 may be particularly useful.

### **Three balloons for two dollars: Developing proportional reasoning, Langrall & Swafford**

Drawing on their own and other published research, this article describes how proportional reasoning problems might be sequenced to develop sophisticated proportional reasoning in children. Four levels of student strategies are described: non-proportional; informal reasoning; quantitative reasoning; and formal proportional reasoning. The authors argue that instruction should support students’ use of informal reasoning strategies to help build toward formal reasoning. The authors also describe four essential components of proportional reasoning: distinguishing proportional and

non-proportional change; identifying situations where using a ratio is reasonable; understanding that the quantities that make up a ratio co-vary in such a way that the relationship between them remains unchanged (is invariant); and unitizing (building per-unit structures to describe increasingly complex relationships).

**Adding it up: Helping children learn mathematics (pp. 241-244), National Research Council**

This book segment briefly summarizes research on proportional reasoning. The segment describes the relationship between ratio and proportion, and introduces the unit rate as a way to compare any two quantities. Statistics are presented on students' poor academic achievement with respect to proportional reasoning. Three important aspects of proportional reasoning are related to student proficiency: making comparisons based on multiplication rather than addition; distinguishing between features of proportional situations that can change and must stay the same; and learning to build composite units. The segment also describes student solution methods (e.g., "building up") and problem types, and presents an argument for attending to the conceptual aspects of proportional reasoning rather than moving directly to a cross-multiplication algorithm.

**Classroom activities for making sense of fractions, ratios, and proportions: 2002 Yearbook, Bright & Litwiller.**

This publication serves as the companion to the NCTM Yearbook *Making Sense of Fraction, Ratios, and Proportions*, and includes classroom activities designed to illustrate Yearbook ideas. The Cocoa and Lemonade Mix activities are similar to the mixing juice problem at the beginning of the toolkit. The John's School activity encourages students to demonstrate an understanding of a proportional relationship by finding the missing dimension of similar figures. The Make a New Puzzle activity focuses on encouraging multiplicative rather than additive comparisons in a situation where the scale factors are not whole numbers. The Centimeters and Inches activity highlights the concept of unit ratio/ rate and helps students connect the idea of constant rate and measurement to proportionality and slope.

**Developing students' proportional reasoning: A Chinese perspective, Cai & Sun.**

The intention of the article is to describe the Chinese perspective on developing students' ability to reason proportionally, which is begun in the elementary grades (in contrast to the standard U.S. introduction of the concept in the middle school grades). Ratio is taught as comparison of quantities using multiplicative relationships. Students learn about and see how the ratio  $a:b$ , the division  $a \div b$ , and the fraction  $a/b$  are related. (The division  $a \div b$  is the operation of the ratio of  $a:b$ , and the fraction  $a/b$  is the result of the ratio operation. This ratio operation  $a/b$  is called the value of the ratio of  $a:b$ .) The article also describes how a scale drawing is used to bridge the concepts of ratio and proportion, and defines proportion as two ratios that have the same value. Ways of developing student understanding of ratio and proportion are addressed, including having students study examples of direct and inverse proportionality in multiple contexts, think

about invariant and covariant problem quantities, study multiple examples, and use multiple solution methods to help them make connections among instructional tasks.

**Improving instruction in rational numbers and proportionality, Smith, Silver, & Stein**

This book is part of a three-volume set designed to support teachers to improve their mathematics instruction. This volume includes 4 cases of teaching episodes around rational numbers and proportionality, the teachers' perspectives on these episodes, mathematics activities, sample teacher responses to the mathematical activities, and facilitation chapters designed to help to identify the key mathematical and pedagogical ideas embedded in the cases. The cases are useful for elucidating key issues of practice, such as the solution methods and representations used by students to solve proportional reasoning problems and how teaching supports or inhibits learning opportunities. Cases focus on connecting fractions, decimals, and percents; ratio and proportion (equivalent ratios); comparison of similar figures; and use of problems involving ratios and percents.

**Making sense of fractions, ratios, and proportions: 2002 Yearbook, Litwiller & Bright**

This book describes important concepts related to the development of the three title ideas across the elementary and middle grades. The articles are intended to offer insight into student thinking and suggestions on how to develop the concepts and skills associated with each target idea. One focus of the book is the transition from additive to multiplicative reasoning. Tasks to assess student understanding of the given concepts are included. Taken as a whole, the book helps to identify the array of math ideas related to proportional reasoning and how they might be structured, including part-whole comparisons, slope, speed, multiplicative reasoning, percents, and proportion.

**Missing-value proportional reasoning problems, Kaput, & West**

This somewhat dense but rich article contrasts with other research on missing value proportional reasoning problems because it focuses on informal rather than algebraically organized solutions and on problem context and semantics. The authors argue that informal reasoning rooted in counting and unit formation influence later student approaches to solving proportional reasoning problems, and they offer an explanation of processes associated with informal reasoning as well as empirical evidence to support their examination of student reasoning.

**More: In depth Discussion of the Reasoning Activities in “Teaching Fractions and Ratios for Understanding,” Lamon.**

This companion book to Lamon's Teaching Fractions and Ratios for Understanding presents and discusses solutions to the sample problems included in each of the chapters.

### **Proportional reasoning: A review of the literature, Tournaire & Pulos**

This article presents a literature review of research on proportional reasoning. It describes strategies used to solve proportion problems, and factors that influence student performance on proportion problems (including problem context, the presence of integer ratios, and numerical complexity). Types of problem tasks (physical, rate, mixture, and probability) and associated student challenges (e.g., the greater difficulty of mixture problems) are discussed, as are successful and unsuccessful strategies for solving proportion problems and a developmental sequence for proportional reasoning strategies.

### **Rational Number Project, Behr, Cramer, Harel, Lesh, & Post (P.I.s)**

The website (<http://www.education.umn.edu/rationalnumberproject/>) contains most research publications and “think pieces” by researchers affiliated with the Rational Number Project (RNP). RNP is “...the longest lasting federally funded cooperative multi-university research project in the history of mathematics education, continuously funded by NSF” from 1979-2002. Most RNP research concerns the learning and teaching of rational number concepts including fraction, decimal, ratio, indicated division, measure and operator, and the components of proportional reasoning. The project has focused on the contributions of multiplication and division understandings to the mentioned rational number concepts, as well as design of effective professional development and appropriate assessment.

### **Teaching multiplication: An analysis of elementary school mathematics teachers’ manuals from Japan and the United States, Watanabe**

The article describes the initial treatment of multiplication in four commonly used textbook series from Japan and the United States. Although there were many common features across the textbooks of the two countries, several features characteristic if the Japanese texts were noted, including the distinction between multiplier and multiplicand; the delayed treatment of 1 and 0 as factors; the concept of *bai* (“times as much/ many”); and the use of horizontal notation during the introductory discussion of multiplication.

### **Young Mathematicians at Work (Book and CD-ROM), Cameron et al., Dolk & Fosnot**

This book and accompanying CD describe classroom events related to a task asking students to compare two situations to determine which is the best buy for cans of cat food. Multiple methods of student thinking are shown (e.g., finding and comparing a unit price, finding a common denominator in order to compare each store’s prices for that quantity of cans), and pedagogical issues related to setting up the problem (e.g., choice of numbers) and moving on after students generate results are described. The student solutions and classroom events support a conversation about equivalence and an exploration of the ratio table. The facilitator’s guide offers suggestions for facilitation of professional development and use of the interactive CD-ROM. The guide also offers instructional suggestions about posing questions, facilitating dialogue, and developing a community. One of the noteworthy features of this resource is the authentic student work and thinking reprinted in the book and shown on the video.