Algebra in the K-5 Classroom: Learning Opportunities for Students and Teachers[1]

By Deborah Schifter, Virginia Bastable, Susan Jo Russell, Lisa Seyferth, and Margaret Riddle

For most Americans, the identifying feature of algebra is the formal equation consisting of variables and signs for the operations and equality. However, beneath the high abstraction of equations like $a(b+c)=ab+ac$ lie ways of reasoning about how quantities can be decomposed and recombined under different operations—ways of reasoning, unlike the conventions of the notation itself, fully accessible to elementary-aged students.

For the last decade, a number of groups have been studying how early algebraic thinking can be introduced into the K-5 classroom (Ball & Bass, 2003; Bastable & Schifter, 1995; Carpenter et al., 2003; Carraher et al., 2000; Kaput & Blanton, 1999; Kaput, 1999; Schifter, 1999, in preparation; Schifter, et al., in press; Smith, 2003). During this time, the first three named authors of this paper have directed projects with groups of teacher-collaborators to investigate what early algebra can mean to their students. Our data consist of cases, written by these teachers, intended to capture their students' thinking as it finds expression in classroom process. The teachers' own reflections—insights, questions—are equally revealing and valued as such.

Together we have found that as children learn about the four basic operations—understanding the kinds of situations the operations can model, sorting out various means

[1] This work was supported by the National Science Foundation under Grant No ESI-0095450 awarded to Susan Jo Russell at TERC and Grant No. ESI-0242609 awarded to Deborah Schifter at the Education Development Center. Any opinions, findings, conclusions, or recommendations expressed in this chapter are those of the authors and do not necessarily reflect the views of the National Science Foundation.

of representing them, and figuring out how to compute efficiently—they observe and comment upon regularities in the number system. For example, they may notice that the calculations 72-38 and 74-40 produce the same result, or that successive answers to a series of problems (10+1=?, 10+2=?, 10+3=?, …) increase by 1. In our view, such regularities, emerging naturally from children's work, become the foundation not only for exploration of generalizations about number and operations, but also of the practices of formulating, testing, and proving such generalizations—and it is these practices that are at the heart of what we mean by "early algebra."

Through our work with teachers, we have found evidence that students' engagement with early algebra can translate into greater computational fluency. Indeed, our collaborators report that these algebraic practices—stating generalizations about the number system and proving them—support *all* students: challenging those who tend to be ahead of their classmates, even as these same practices help struggling students gain access to basic arithmetic principles.

Our collaborators' cases also form the core of a professional development seminar designed to help teachers develop the knowledge and skills needed to support early algebraic reasoning in their classrooms (Schifter, et al., in preparation). By studying such cases, teachers come to understand the importance of generalization and the central role visual representations of the operations play in developing arguments for infinite classes of numbers.

Furthermore, we have found that when teachers study how elementary school students think algebraically, they are themselves provided a context that endows the formal notation with meaning, hence utility. Starting with an *idea*, a generalization, instantiated with particular numbers or expressed in natural language, teachers can learn the conventions of algebraic notation to express that generalization concisely. This contrasts with the experience of many elementary teachers who learned the syntax of algebraic notation in high school, but without meaning.

In this paper, we present two classroom cases, one drawn from kindergarten, the other, from fourth grade, to illustrate how early algebraic thinking can arise quite naturally in the context of instruction on number and operations. Then we present scenes from a professional development seminar to illustrate some of what teachers can learn from studying such cases.

Kindergartners as Algebraic Thinkers: Adding the Same Amount to Unequal Amounts

Kindergarten teacher, Lisa Seyferth, had set up her students to play, Double Compare[2], a card game similar to War: Each card bears a numeral, from 1 to 6, and a picture of that number of objects. Players lay down the top two cards from their piles, and the player with the higher total when the numbers on the cards are combined says "me." For example, when Wei[3] turns over a 2 and a 6, and Marta turns over a 3 and a 4, they count up the totals and Wei says "me."

No sooner was the game in question underway, than Seyferth realized there were several pairs of children saying "me" or "you" before they could possibly have had time to find the sum of their numbers—but they were always right! For example, Martina had 6 and 2 and Karen had 6 and 1, Karen quickly said "You." When Seyferth asked how she

[2] The game, Double Compare, is taken from *How Many in All?,* a kindergarten unit of *Investigations in Number, Data, and Space* (TERC, 1998).
[3] Student names are pseudonyms

knew, Karen pointed to the 2 and said, "This is big. Even though these are the same [the 6s], this [the 6 and 2] must be more."

Paul and his partner had similar sets of hands. Paul put down 6 and 3 and his partner, 6 and 1. Paul said, "I had 6 and he had 6, and then I had a higher number." When Seyferth asked what their cards added up to, each counted up all the little objects pictured on his cards to get the totals. But as they continued to play, they frequently did *not* count to accurately determined who got to say "me." Thus on a later turn, Paul had 4 and 3 and his partner had 4 and 5. Paul's comment was, “I have 3 and he has 5." He knew he could ignore the two 4s.

Seyferth had set up a task with the intention of giving her kindergartners an opportunity to practice counting up to find totals. But it seemed her students were subverting her goals for the exercise: They could play many rounds correctly without doing any totaling.

However, Seyferth went beyond noticing what her students *weren't* doing, to figure out what they *were* doing. Implicit in their moves was a general principle: If each child has a card of equal value, compare the other two cards; the child with the card of greater value has the greater total. When it was time to put the cards away, Seyferth brought the students together to discuss this idea. She later wrote,

> We talked for a while about how all the pairs “ignored” cards when each partner had the same value, and only paid attention to the cards that were different. Martina said that 6 and 3 is more than 6 and 1 because the 3 is bigger than the 1. I asked, "What about the sixes?" and she said, "They're the same." Paul added, "They don't matter. You don't have to pay attention to the sixes." I put out a few

more sets of cards, varying the number that was the same. ("Does this only work for 6?" "No.") They said it always works, and Paul reiterated that you don't have to pay attention to the numbers that are the same.

As young children begin to learn about numbers and operations, they begin to see regularities in our number system, and to think about what stays the same among things that are changing. The regularity, or generalization, illustrated above—if one number is greater than another, and the same number is added to each, the first total will be greater than the second—is one example. This statement is true for *any* three numbers. For example, since $54 > 36$, $54 + 98 > 36 + 98$. This idea can be expressed in algebraic notation: *If a, b, and c are numbers and if* $a > b$, *then* $a + c > b + c$.

However, it is important not to attribute too much to these kindergartners. The children are playing a card game that involves the numbers 1 through 6. They determine who gets to say “me” by combining the number of pictured objects when they each turn over two cards. We do not know if the children have a conception of numbers greater than 6, whether they think in terms of the operation of addition, or whether the ideas they are discussing apply to contexts outside the card game.

Yet as this vignette illustrates, teachers can invite even young children to talk about the general ideas implicit in their actions. Consider Seyferth’s moves: First, she noticed that some of the children were playing the game in a way that indicated they may have been acting on a general principle or rule they had noticed but had not articulated. Next, she decided to bring this to the attention of the whole class during discussion. Then she asked her students to explain their rule and why it worked by suggesting additional examples for them to consider. Finally, she posed the question: Does this rule work for just a few examples or does it apply more generally?

This discussion provided the students, who had been acting on an unstated rule, the opportunity to express their thinking in words and to begin to see what kinds of examples the rule applies to and why it works. In addition, students who had not noticed the regularity were given a demonstration that this kind of thinking—looking across examples for general principles—is part of mathematics class routine. By posing the question, "Does this work only for 6?" the teacher brought into her classroom the practice of asking, "Will it always work?"

As the year progressed, Seyferth continued to look for opportunities to encourage her students to think in terms of general principles. For example, when she laid out a set of 8 checkers for a counting exercise—3 red and 5 black—she asked, "Does it matter if we count the red checkers first and then the black, or start with the black and then count the red?" When her students agreed that it didn't matter in what order you counted, as long as you didn't take any away or add any more on, she pushed the class to be clear about the claim they were making: "Is this true only about checkers, or does it work if we have 3 yellow teddies and 5 blue teddies? Is this something special about the number 8, or would it work with other numbers?"

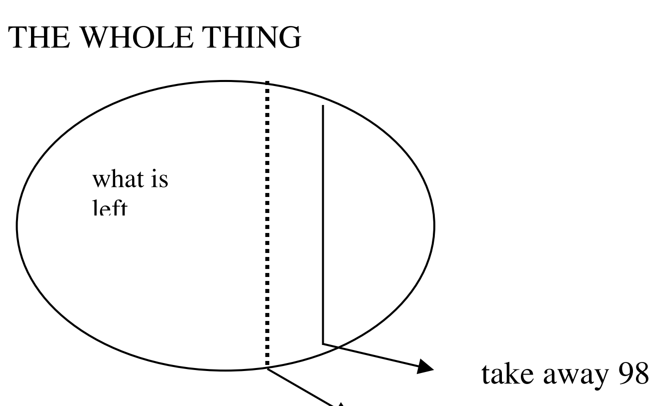
Even while children are still learning to count, discussions like these lay the foundation for understanding the logic of the operations. Thus, the children will soon be working with the operation of addition. Just as the insights from Double Compare can be called upon as students compare sums, the corresponding question about counting checkers might be: Does it matter if we add the red checkers first and then the black, or if we start with black and then red? That is, they are laying the foundation for articulating the commutative property of addition: $a + b = b + a$.

Fourth Graders as Algebraic Thinkers: Subtracting Less Results In More

As part of her morning routine, Margie Riddle had set her class some subtraction problems. Included among them were 145-100 and 145-98. As the children began to consider the latter problem, Riddle realized that here was an opening that held much potential for learning, and so she deferred further discussion until the math lesson later in the day.

What Riddle had seen was this: Many of the children realized there was a connection between the two problems. However, after calculating 145-100=45 but *before* actually solving 145-98=?, they weren't sure if the answer to the second problem would turn out to be 2 more or 2 less than 45. Once they did solve it and knew the answer was 47, they wondered, why was it 2 *more*? After all, when they changed 100 to 98, they *subtracted* 2, so why *add* 2 to get the right result? Their puzzlement, Riddle felt, could lead her students to a deeper appreciation of the meaning of the operation of subtraction.

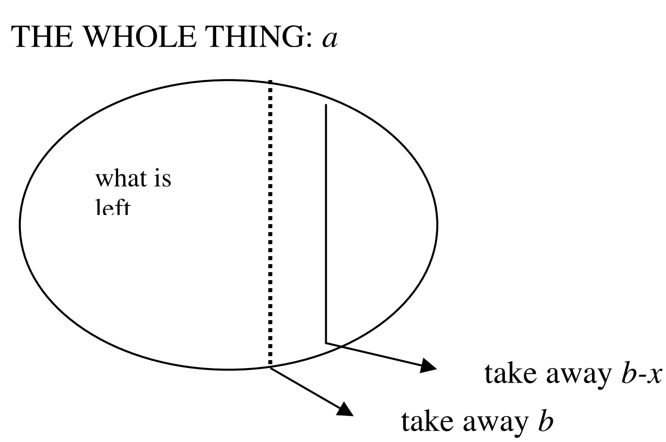
When the class returned to this question later that day, Riddle's students began, as was their habit, by sharing a variety of ways to calculate 145-98. But at a certain point in their discussion, Brian insisted on explaining his thinking to his classmates. He struggled to find the words. "It goes with the problem before," he declared. "It's like you've got this big thing to take away and then you have a littler thing to take away so you have more. Can I draw a picture?" He went up to the blackboard, thought for awhile, and then drew a blob:



Riddle described what happened next. “Brian’s classmates were watching and listening fairly intently, and suddenly, inspired by his presentation, Rebecca said excitedly, ‘Yeah, it's like you have this big hunk of bread and you can take a tiny bite or a bigger bite. If you take away smaller you end up with bigger.’” Riddle asked if she thought this would always be true, and Rebecca said, “I think so.”

To this point Max had been quiet. But now, inspired in turn by Rebecca's explanation and Brian's picture, he carried the unfolding line of thinking even further: "Yeah, the less you subtract, the more you end up with. AND ," he continued with great emphasis, “the thing you end up with is exactly as much larger as the amount less that you subtracted."

Max’s insight, translated into algebraic notation, would be $a – (b – x) = (a – b) + x$ (where, a, b, and x are positive numbers, $a > b > x$). Referring back to the diagram, when the strip between the dotted line and the solid line (x) is subtracted from region to the right of the dotted line (b), it is joined to the region to the left of the dotted line ($a – b$). However, it is not the notation, but the *generalization*, i.e, the *idea* expressed by this equation, that qualifies this fourth-grade discussion as algebraic.



Consider how the teacher moves through this episode. By posing the related problems (145 – 100 = ? and 145 – 98 =?), Riddle had created a situation in which her students, while working on a subtraction exercise, could notice a deep feature of the operation of subtraction itself. As she listened to her students' expression of puzzlement, she decided to center her next mathematics lesson on their confusion.

Already established in this classroom was the practice of representing a mathematical situation with such aids to thinking as diagrams, story contexts, or manipulatives. So when Brian offered a diagram of the problem situation, other students were ready to seize on it to articulate their own thoughts. Riddle picked up on their thinking to get at a general principle: "Rebecca, do you think this is always true?"

The work of generalizing and justifying in the elementary classroom has the potential of enhancing the learning of *all* students. Teachers with whom we have collaborated have realized this potential in their classrooms (Schweitzer, in press). They report to us that students who tend to have difficulty in mathematics become stronger mathematical thinkers through this work. As one teacher wrote, "When I began to work on generalizations with my students, I noticed a shift in my less capable learners. Things seemed more accessible to them." When generalizations are made explicit—in natural language and/or through spatial representations used to justify them—they become accessible to more students and can become the foundation for greater computational fluency. Furthermore, the disposition to create a representation when a mathematical question arises supports students in reasoning through their confusions. Brian, who

Riddle later explained was a tentative learner in mathematics, created the representation that illuminated an important idea for the whole class.

At the same time, students who generally outperform their peers in mathematics find this content challenging and stimulating. For them, the study of number and operations can extend beyond efficient computation to the excitement of making and proving conjectures about mathematical relationships that apply to infinite classes of numbers. As one teacher explained, "Students develop a habit of mind of looking beyond the activity to search for something more, some broader mathematical context to fit the experience into." Max, one of the most mathematically successful students in Riddle's class, listened carefully to his classmates and then enjoyed the challenge of formulating a precise statement of the generalization.

Teachers as Algebraic Thinkers: Algebra as the Expression of Ideas

To encourage algebraic thinking among their students, teachers must recognize the importance of discussing mathematical generalizations. And so, self-evidently, must they be capable of noticing when students' actions and observations broach such. Furthermore, they must have a sense of the kinds of problems or questions likely to stretch student thinking, and they must be able to steer student discussion in productive directions. Cases of classroom episodes—like those by Lisa Seyferth and Margie Riddle from which the examples presented above were drawn—can be one mechanism by which teachers can begin to develop these skills.

But it is also important for elementary teachers to understand more deeply the mathematics content their students will encounter as they pass *beyond* the elementary grades. What may not be as obvious is that these K-5 cases can provide rich contexts for such understanding: Rather than (re-)encountering algebraic manipulation as syntax devoid of meaning, teachers here learn to use algebraic notation to represent *ideas*.

In our work with elementary teachers, we frequently ask them about their prior experience with algebra. Generally, their responses fall into one of two categories: 1) some never understood the purpose of algebraic notation and generally felt hopelessly lost, and these report that the mere sight of algebraic notation makes their stomach clench; 2) for others, their study of algebra was pleasurable because it felt like a puzzle to them, and they were adept at finding solutions. But in neither case do these teachers say they looked for meaning in the algebra problems they worked on. Thus, for all of them, to begin with an idea, which may be expressed in natural language and/or for which they can offer examples, and *represent it using algebraic notation* is a large and novel step.

Teachers Discuss the Kindergarten Case

The following scene is drawn from a teachers' seminar[4] in which participants were working from Lisa Seyferth's case. The session began with attempts to state in English the rule underlying her Kindergartners' actions in the Double Compare exercise.

Antonia started the group off with the suggestion, "If two cards are the same and two cards are different, then the one with the larger of the different numbers says 'me.'"

Taking a moment to think, Madelyn expressed doubt about Antonia's generalization. "I don't know if this is too picky or not—but what if one person has the two numbers that are the same? Like, what if one person has 5 and 5, and the other person has 6 and 2? It satisfies the conditions, but it's wrong. The person who has 6 doesn't win."

[4] These vignettes are composites derived from field notes taken while piloting *Reasoning Algebraically about Operations*. Teachers' names are pseudonyms.

M'Leah suggested rewording Antonia's statement: "If each person has two cards, and one of the cards each person has is the same, then the person with the higher different card says 'me.'" Madelyn said this wording worked for her, and others nodded.

Lorraine offered a different version. "A number plus a big number is more than a number plus a small number." The group then talked about some of the assumptions in this statement. Lorraine says "a number" twice, but it must be the same number. And when she says "a big number" and "a small number," she means in relation to one another. For example, if you have the cards 5, 3 and 5, 1, then 3 is the big number, even though it isn't the biggest number in the two hands. In this case, it's called the big number because it's compared to 1. And if you have the cards 2, 3 and 2, 6, then 3 is the small number, even though it isn't the smallest number in the two hands. Here, the 3 is small because it's compared to 6.

The teachers were struck by how difficult it was to state the generalization in English without ambiguity. But rather than push for a less ambiguous version, the facilitator chose to compare the two statements that had been offered.

> If each person has two cards and one of the cards each person has is the same, then the person with the higher different card says "me."
>
> A number plus a big number is more than a number plus a small number.

As the teachers discussed the differences between the two statements, two major points emerged: 1) One statement was about cards, the other about numbers. 2) Only one statement was about the operation of addition.

This is a very important issue to keep in mind when working with children. It might *seem* that students grasp a generalization, but when confronted with the same mathematical idea in a different context, they fail to recognize it. Thus, the generalization the kindergartners are working on might not extend beyond the card game.

In this session, one teacher observed, "The same thing comes up in older grades, too. You work on an idea in one context and you think the idea is really solid for the kids, but when you change the context, you realize the idea has disappeared."

The facilitator now objected that the idea may still be present. "Rather, the idea they had in the first place was not as general as you thought. But the idea they had, confined to the original context, is likely to still be there, and it's something you can draw upon as you work to help them understand the more general idea. Indeed, this is precisely why it's important to have discussions about these ideas. By articulating them, making them part of the shared knowledge of the class, they can be referred to in future discussions. You can ask, 'Do you remember when we talked about such-and-such? Is that anything like what we are discussing now?'"

Having considered the *idea* the kindergartners were working on, the teachers were ready to represent it in algebraic notation. Grace presented her version first: $a + (>b) > a + (<b)$. She explained, "I wasn't really sure how to write it, but I decided to take a stab at it. If you add to a some number larger than b, the sum will be more than if you add to a some number smaller than b. That's what I was trying to write."

As children will invent their own spelling as they learn to write, so, too, Grace invented her own notation in her attempt to express the idea under discussion. Now she, and her classmates, had the opportunity to see which aspects of her statement did not observe the generally accepted conventions, and to write a statement that did.

The facilitator explained to Grace, "Even though I can see what you mean, mathematicians don't write '>*b*' to mean some number greater than *b*. They might write '$x>b$,' which means that *x* is some number greater than *b*."

Then another teacher, Mishal, offered two ways to state the generalization. "You've got three numbers, *a, b,* and *c*. You can write, If $a > b$, then $a + c > b + c$. And you can write, If $a > b$, then $c + a > c + b$."

Risa said that the second line looks just like the second of our statements: "A number plus a big number is more than a number plus a small number." Denise pointed out that it's much easier to read the algebraic notation. "You don't have to go through all the talk we did about what 'a number' and 'big number' and 'small number' mean." Leeann said that algebraic notation is easier only if you already know how to read it. "Right now, I still need the English to help me."

Although the algebraic statements that were now before the seminar were correct, the facilitator recognized there was more for the teachers to think about. She asked participants to go in the other direction, to look at the algebraic notation with particular numbers in mind. "What if we had these cards: 4, 2 and 4, 3? What are the values of *a, b*, and *c*?"

June answered, "*c* is 4, *b* is 2, *a* is 3."

The seminar group worked through several possible sets of cards without difficulty until they came to 3, 3 and 5, 3. At this point, there was some confusion—there were too many 3s!

Denise, who was already quite fluent in algebraic notation, pointed out that this didn't matter; *a* is 5, and *b* and *c* can both be 3. The facilitator had the group go back to the English language version of the statements to see if they covered this case. Once the group was satisfied they did, they went back to algebraic notation. The facilitator explained that when *c* appears twice, it must refer to the same number in both places. But if different letters are used, the numbers they stand for need not be different.

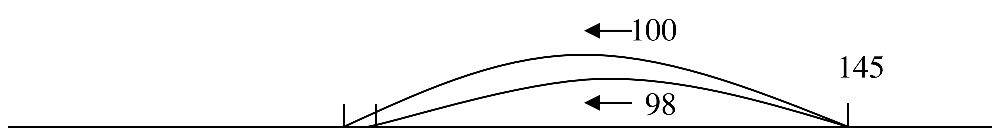
Finally, the facilitator asked about 4, 2 and 6, 3. Everyone was pretty clear that this example wasn't covered by the generalization they were working with because all four numbers are different. But Madelyn pointed out that they could write out a statement that covered this case, too: If $a > b$ and $c > d$, then $a + c > b + d$.

Teachers Discuss the Fourth-Grade Case

Although the teachers found it a challenge to articulate precisely the generalization implicit in the kindergartners work, they began with a strong sense of what the idea was. However, when they read Margie Riddle's case, they needed to work hard to get hold of the idea those fourth graders were working on. Kaneesha said, "I can look at Brian's picture, but at first I wasn't sure I got it. It really took Rebecca's comment about the bread to help me see it. If you have a hunk of bread and take a small bite, you end up with more than if you take a large bite."

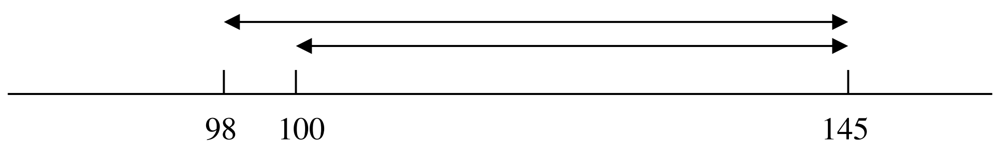
Jorge agreed that it was Rebecca's image that helped him. "When she puts it like that," Jorge said, "it's so clear what's happening with the numbers. Whenever you see a subtraction problem, you can think about breaking off a hunk of bread. If you take less, you leave more. Wow."

Risa said that it was easier for her to see on the number line than on Brian's blob:



Risa explained, "If you start at 145 and go back 100, you end up farther to the left than if you go back 98."

Charlotte said she sees subtraction as distance between two numbers. "When you look at the number line, you can see the distance between 98 and 145 is greater than the distance between 100 and 145. So 145-98 is more than 145-100."

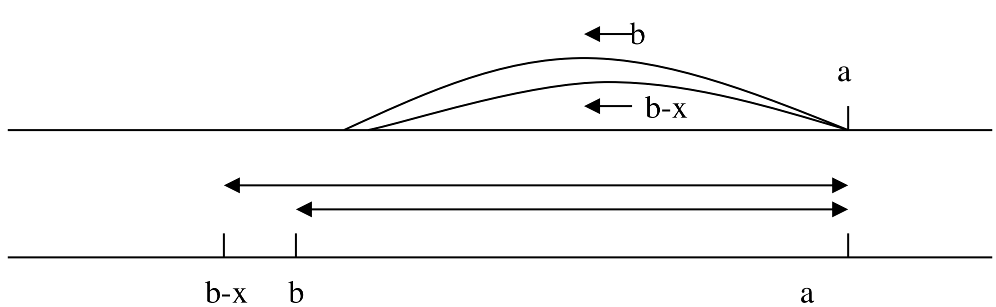


Next the facilitator asked the class which of the two representations they preferred, Risa's or Charlotte's, and it seemed everyone had a preference. Then she challenged the teachers to spend a few minutes making sense of the representation they had not favored.

The facilitator asked both Risa and Charlotte, could they change their representations to illustrate the generalization? Risa said, "Sure, all I have to do is erase the numbers. I can start at any place—that's the number I'm subtracting from. If I jump back more, I end up further to the left than if I jump back less."

Charlotte looked at her number line for a while, and then said it works for hers, too. "Like Risa did, fix the number you're subtracting from. If you start at a smaller number, you have to go farther to reach your destination than if you start at a larger one."

Each participant changed her representation: "a" replaced 145; "b", 100; and "b-x," 98.



When the facilitator asked for a representation of the statement in algebraic notation, Mishal suggested, “If $a - b = c$, then $a - (b - x) = c + x$.” Since the teachers had explained that, once an algebraic statement was made, it helped to translate it back into specific numbers, the facilitator gave them two sets of numbers to test. “What if $a = 75$, $b = 25$, and $x = 3$? if $a = 149$, $b = 18$, and $x = 10$?”

Once the teachers felt confident they understood Mishal’s statement, the facilitator suggested another way to represent the same idea. “Mishal’s way of writing out the idea is correct, but it’s also possible to write it as a single equation:”

$$a - (b - x) = (a - b) + x.$$

Then she explained, “I understand that algebraic notation can become pretty terse. When we see the idea written out this way, what we need to understand is that the expression ‘$(a - b)$’ represents at once the operation of subtraction, $a - b$, and the single number that results from the act of subtraction. Mishal used ‘c’ to represent that single number.”

This was not the first time the facilitator introduced the idea of the duality in the notation. Earlier, when working on a case where students discussed what happens to a sum when 1 is added to one addend, the teachers represented this idea as, If $a + b = c$, then $(a + 1) + b = c + 1$. The facilitator suggested this same idea can be written as $(a + 1) + b = (a + b) + 1$, and explained that ‘$(a + b)$’ at once represents the operation of addition, $a + b$, and resulting sum.

Grace commented, “You know, in kindergarten, children think of 4 + 2 as simply 4 + 2; we need to teach them that, when we think about the total, we say that it’s 6: 4 + 2 = 6. It feels like there’s something connected here with the algebraic notation, but it’s backwards. We don't need to specify the total with a single symbol. It's enough to leave it as $a + b$."

Returning to discussion of the Riddle's case, Denise came to the board to explain her way of thinking about the equation. As she wrote, she said, "When I took algebra in high school, I learned to distribute the subtraction sign, like this:"

$$a - (b - x) = a - b - (-x).$$

Denise continued, "We also learned $-(-x) = x$. So you get this equation:"

$$a - (b - x) = a - b - (-x) = a - b + x.$$

She concluded, pointing to $a - (b - x) = (a - b) + x$, "And that's how you get to this. The thing is, now I'm looking at that same equation with a completely different sense of what it means. I don't have to think about distributing the subtraction sign, and I don't need to think about $-(-x)$. It feels really different to think, the less you subtract, the greater the result. *That's* what the equation says."

The facilitator specified more precisely, "That's what the equation says as long as x is a positive number. We'll soon extend the domain of number to integers, and then we'll need to look back at this statement."

This seminar session was designed to achieve a number of purposes and to that end the two cases were being considered together.[5] First, teachers worked to understand the ideas of the students in the terms the students themselves use. Next they analyzed the idea implicit in the kindergartners' card-game strategy and considered different levels of generality of that idea—from a claim that applied only to this particular card game to a claim about the operation of addition itself. In order to understand more deeply the idea

[5] Omitted from the episodes presented in this paper are discussions of pedagogy.

the fourth graders were working on, the teachers carefully read the students' discussion and then represented the same ideas on a number line.

Finally, the teachers worked to express these ideas in the language of algebra. This provided them the opportunity to experience the power, precision, and economy of algebraic notation while, at the same time, becoming familiar with its conventions. After thinking through the *ideas,* they express them as algebraic identities or inequalities. In sum, the teachers have learned to use the notation meaningfully, and the conventions governing its syntax can be seen to make sense.

Conclusion

Alarmed by the difficulty many secondary students have learning algebra, policy makers have been led to consider introducing "early algebra" into the elementary grades. In the last decade, some researchers have taken on the challenge of investigating what algebraic thinking might look like so introduced and how young students might engage in it. The authors of this paper, together with a group of elementary teachers, have identified generalizations that arise quite naturally from students' work on number and operations and explored what happens when teachers make explicit these generalizations for their students to consider. They challenged their students with the questions, Will this work for all numbers? and How can we decide? The Kindergarten and fourth-grade cases presented in this chapter provide examples of how this work can be situated in elementary classroom contexts. Teachers have found that consideration of such generalizations, especially when grounded in visual representations students can use to explain them, support the work of all their students, those who tend to struggle as well as those who tend to excel.

To realize such practices in the classroom will require teachers to learn much. To that end, we argue that, rather than enrolling elementary teachers in a conventional high school or college algebra course, they should learn about algebra and its connections to arithmetic by working with mathematical ideas based in the K-5 classroom. This approach, which the authors have been developing over the past several years, allows teachers to internalize images of what it means to engage their students with algebraic ideas, identify those ideas that lead to fruitful exploration at different grades, and consider the kinds of tasks and questions likely to elicit them. At the same time, as this paper shows, teachers can learn how these ideas are expressed in algebraic notation. Working with K-5 cases supports teachers' attempts to learn algebra with meaning and to see algebra as a seamless strand that begins in Kindergarten and extends into secondary school and beyond.

However, even strong teaching in elementary school will not by itself eliminate the difficulties secondary students encounter in learning algebra. If secondary teachers are to mine their students' deep understanding of the number system developed through exploration of early algebraic ideas in the elementary grades, they, too, will need to learn to teach algebra not only syntactically, but also as an expression of ways of reasoning about how quantities can be decomposed and recombined under different operations.

References

Ball, Deborah L., & Bass, Hyman. "Making Mathematics Reasonable in School." In *A Research Companion to the Principles and Standards for School Mathematics*, edited by Jeremy Kilpatrick, W.Gary Martin, & Deborah Schifter, pp. 27-44. Reston, VA: National Council of Teachers of Mathematics, 2003.

Bastable, Virginia & Schifter, Deborah. “Classroom Stories: Examples of Elementary Students Engaged in Early Algebra. In *Employing Children’s Natural Powers to Build Algebraic Reasoning in the Context of Elementary Mathematics*, edited by J. Kaput, M. Blanton, & D. Carraher. Mahwah, NJ: Lawrence Erlbaum Associates, in press.

Carpenter, Thomas P., Franke, Megan, & Levi, Linda. *Thinking Mathematically*. Portsmouth, NH: Heinemann, 2003

Carraher, David, Brizuela, Barbara & Schliemann, Analucia. “Bringing out the algebraic character of arithmetic: Instantiating variables in addition and subtraction.” In *Proceedings of the 24th Conference of the International Group of the Psychology of Mathematics Education. Vol 2*, edited by Te Nakahara & Me Koyama, pp. 145-151. Hiroshima, Japan: Hiroshima University, 2000.

Kaput, James & Blanton, Maria. “Algebraic reasoning in the context of elementary mathematics: Making it implementable on a massive scale.” Paper given at the annual meeting of the American Educational Research Association, Montreal, April 1999.

Kaput, James. “Teaching and Learning a New Algebra.” In *Mathematics Classrooms that Promote Understanding*, edited by E. Fennema & T.A. Romberg, pp. 133-155. Mahwah, NJ: Lawrence Erlbaum Associates, 1999.

Schifter, Deborah. “Reasoning about Operations: Early Algebraic Thinking, Grades K through 6.” In *Mathematical Reasoning, K-12:* 1999 Yearbook of the National Council of Teachers of Mathematics (NCTM), edited by Lee Stiff and Frances Curio, pp. 62-81. Reston, VA: NCTM, 1999.

Schifter, Deborah. "Proof in the Elementary Grades." In *The Teaching and Learning of Proof Across the Grades,* edited by Despina A. Stylianou, Maria Blanton, & Eric Knuth, in preparation.

Schifter, Deborah, Bastable, Virginia, Russell, Susan Jo, & Monk, Stephen. *Reasoning Algebraically about Operations*. In preparation.

Schifter, Deborah, Monk, Stephen, Russell, Susan Jo, & Bastable, Virginia. "Early Algebra: What Does Understanding the Laws of Arithmetic Mean in the Elementary Grades?" In *Employing Children's Natural Powers to Build Algebraic Reasoning in the Context of Elementary Mathematics*, edited by James Kaput, Maria Blanton, & David Carraher. Mahwah, NJ: Lawrence Erlbaum Associates, in press.

Schweitzer, Karen. "Teacher as Researcher: Research as a Partnership." In Teachers Engaged in Research: Inquiry into Mathematics Practice, Grade K-2, edited by Stephanie Z. Smith and Marvin Smith (Denise S. Mewborn, series editor). Reston, VA: NCTM, 2006.

Smith, Erick. "Stasis and Change: Integrating Patterns, Functions and Algebra Throughout the K-12 Curriculum." In *A Research Companion to the NCTM Standards*, edited by James Kilpatrick, W. Gary Martin, & Deborah Schifter. Reston, VA: NCTM, 2003.

TERC. *Investigations in Number, Data, and Space*. Chicago, IL: Scott Foresman, 1998.