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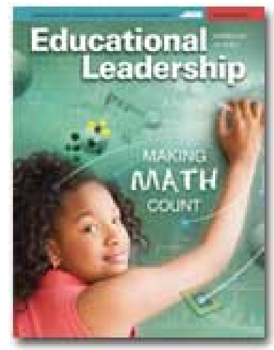

What's Right About Looking at What's Wrong?

Deborah Schifter

Both students and teachers gain new mathematical understanding by examining the reasoning behind a student's incorrect answer.

To teach mathematics for conceptual understanding, we need to treat it primarily as a realm of ideas to be investigated rather than a set of facts, procedures, and definitions to be used. To implement the former approach, teachers must have a deep understanding of content as well as the skill to implement concept-based pedagogy. And these greater demands on teachers, in turn, require well-thought-out forms of professional development. The following classroom lesson illustrates some of the issues involved.

Going Beyond Procedures

Liz Sweeney's 5th grade students all knew the standard procedure for multiplying multidigit numbers. On the day when a research team from the Education Development Center videotaped her class,[1] however, Ms. Sweeney wanted her students to go beyond the procedure. She asked them to find at least two ways to determine the products of several multidigit multiplication problems.

The students worked on this challenge, meeting in small groups to talk about their strategies. With just a few minutes left at the end of the period to discuss their work as a whole group, Ms. Sweeney asked Thomas to write his strategy for solving one of the problems (36 × 17) on the board, even though it was incorrect. Thomas wrote

$$36 + 4 = 40$$
$$17 + 3 = 20$$

$$\begin{array}{r} 40 \\ \times\ 20 \\ \hline 800 \\ -\ \ 4 \\ \hline 796 \\ -\ \ 3 \\ \hline 793 \end{array}$$

Even Thomas knew his answer was wrong. Other strategies had already determined that the answer was 612. But he explained his reasoning to the class: To make the problem easier, he rounded up by adding 4 to 36 and 3 to 17; then he multiplied 40 × 20 to get 800, and subtracted the 4 and the 3 that he had added earlier, getting a final answer of 793.

Ms. Sweeney told the class what she had noticed as Thomas presented this method to his small group:

> So I liked this—I was feeling comfortable with it, and it looked like a good strategy, and it was neat. And then Dima was all antsy in his seat, saying, “That's not what I did and my answer is really different”. . .
>
> So, tonight for your homework, I want you to copy down Thomas's method in your homework books, and I want you to figure out, What was Thomas thinking? And using the first steps of his strategy, how would you revise his approach to come up with a different answer?

The way we learn mathematics

Ms. Sweeney's behavior may puzzle readers whose images of effective teaching derive from the mathematics classrooms of their childhood. For many decades, mathematics has been taught the same way: The teacher demonstrates procedures for getting correct answers and then monitors students as they practice those procedures on a set of similar problems. Why did Ms. Sweeney ask her students, who already knew one efficient way to multiply 36 × 17, to find alternative strategies to do it? Why, at the end of class, did she ask a student to present a strategy that produced an incorrect result? And why did she ask the rest of the class to examine his strategy for homework?

When we view Ms. Sweeney's behavior from an alternative perspective, it becomes comprehensible. She acted on the belief that mathematics is much more than a set of discrete facts, definitions, and procedures to memorize and recall on demand. In her view, mathematics is an interconnected body of ideas to explore. To do mathematics is to test, debate, and revise or replace those ideas. Thus, the work of her class went beyond merely finding the answer to 36 × 17; it became an investigation of mathematical relationships.

implications for instruction

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Where Did Thomas's Error Come From?

This was not the first time Liz Sweeney had asked her students to think about different strategies for calculation. She had been assigning similar exercises for all four of the basic operations. By considering the *action* of the operation, students could develop such strategies independently. For example, when asked to add 18 + 24, students might consider the action of addition as the joining of two sets and devise a variety of methods for decomposing and recombining the addends:

- Decompose 18 into 10 and 8; decompose 24 into 20 and 4; add the tens, 10 + 20 = 30; add the ones, 8 + 4 = 12; add the results, 30 + 12 = 42.
- Take 2 from the 24 and add it to the 18. This becomes 20 + 22, or 42.
- Add 2 to 18 to get 20, 20 + 24 = 44. Then remove the 2 you have added on, 44 - 2 = 42.

The activity of devising calculation strategies and explaining why they work helps students cultivate several important mathematical capacities. Students develop a stronger number sense and become more fluent with calculation. They gain an understanding of place value when they decompose numbers into tens and ones. And they come to expect that mathematics will make sense and that they can solve problems through reasoning.

When Ms. Sweeney asked the class to multiply 36 and 17, Thomas decided to try out a strategy that he had used successfully to *add* two multidigit numbers: round up, perform the operation, and then subtract what had been added when rounding up. Thomas was reasoning by analogy, which is often a fruitful way to approach a problem. In this case, the analogy would not hold. But Thomas *was* reasoning; he was not merely careless.

Thomas's mistake—applying an addition strategy to a multiplication problem—is quite common. When faced with multidigit multiplication, such as 12 × 18, both children and adults frequently try (10 × 10) + (2 × 8). After all, to add 12 and 18, one could operate on the tens, operate on the ones, and then add the total. But multiplication involves a different kind of action, and thus requires a different set of adjustments after the factors have been changed or decomposed.

A Context for Multiplication

To think about the action of multiplication, it is helpful to envision a context in which the calculation might be used. For example, Thomas's classmate James thought of 36 × 17 as 36 bowls, each holding 17 cotton balls. With this context in mind, he could imagine an arrangement of bowls of cotton balls that would lend themselves to calculation.

James explained that first he arranged the bowls into groups of 10. Each group of 10 had 170 cotton balls (10 × 17), and there were three groups of ten (170 + 170 + 170). Besides the groups of 10 bowls, there were another 6 bowls with 17 cotton balls in each (6 × 17). To simplify that calculation, James thought of each bowl as having 10 white and 7 gray cotton balls, which yielded 60 white balls (6 × 10) plus 42 gray balls (6 × 7), for a total of 102 cotton balls in those 6 bowls. Then he added 170 + 170 + 170 + 102, which came out to 612.

A basic mathematical principle underlying James's method is the distributive property of multiplication over addition, which says that (10 + 10 + 10 + 6) × 17 = (10 × 17) + (10 × 17) + (10 × 17) + (6 × 17). The distributive property also says that 6 × (10 + 7) = (6 × 10) + (6 × 7). James knew how to apply the distributive property, but when he worked with an image of cotton balls arranged in bowls, he was not merely manipulating numbers based on a set of rules he had memorized. He was able to perform the calculation as it made sense to him—that is, as it followed from his image of the context.

As Thomas, James, and their classmates developed their strategies in small groups, Ms. Sweeney went from group to group, sometimes asking questions or making suggestions and sometimes just listening. Having observed Thomas's mistaken strategy, she decided that it provided a learning opportunity for the class. When she gave the homework assignment, she was asking her students to go beyond evaluating whether the strategy was correct or not; she was challenging them to determine where it went wrong and how to make it right. To answer that question, students needed to examine closely the difference between addition and multiplication, highlighting the importance of thinking in terms of images like James's. This task also gave them an opportunity to state the distributive property explicitly. This one homework assignment yielded two further days of deep mathematical discussion in Ms. Sweeney's 5th grade class.

Teachers Consider Thomas's Strategy

In a professional development seminar,[2] my colleagues and I explored Ms. Sweeney's approach with a group of teachers. After viewing the video clip, many of the teachers were initially shocked by Ms. Sweeney's behavior. They didn't understand why she would “embarrass a student” by asking him to share his incorrect work. Some were dismayed that she would “punish the class” by assigning homework because one student made an error.

Rather than discuss these issues immediately, the facilitator asked the teachers to examine Thomas's strategy for themselves. After Thomas added 4 to 36 and 3 to 17, what would he need to subtract in order to get the correct result?

The teachers went to work in pairs and threes to examine different ways to approach the problem. The facilitator moved from group to group, listening to teachers, asking them to explain in more detail, and sometimes suggesting an approach. When each group had developed at least one way to think about the problem, the facilitator brought them all together to present their ideas.

Annie volunteered to share her initial thinking, which she realized was not completely correct. She said, “I did something that seems like it should be right, even though I know it's not.” She explained that when Thomas added 4 to 36 and 3 to 17 and then multiplied 40 × 20, he wasn't adding 4 units and 3 units, but 4 groups of units and 3 groups of units. She continued,

> So I first thought you need to subtract 4 groups of 17 and 3 groups of 36. But when I did the calculation, 800 - (4 × 17) - (3 × 36), I got 624—not 612, which we already know is the answer.
>
> I didn't take away enough, so I thought maybe I multiplied by the wrong size group. Maybe I need to take away 4 groups of 20 and 3 groups of 40. But when I did this calculation, 800 - (4 × 20) - (3 × 40) = 600, I ended up with an answer that was too small!
>
> I thought that was really strange. Then the facilitator came and suggested that we think of a story context.

A story context would allow the teachers to picture the steps of the problem, as James had done. Ming suggested the following context:

> There are 40 children in a class, and they each paid $20 for a field trip. The teacher collected 40 ×

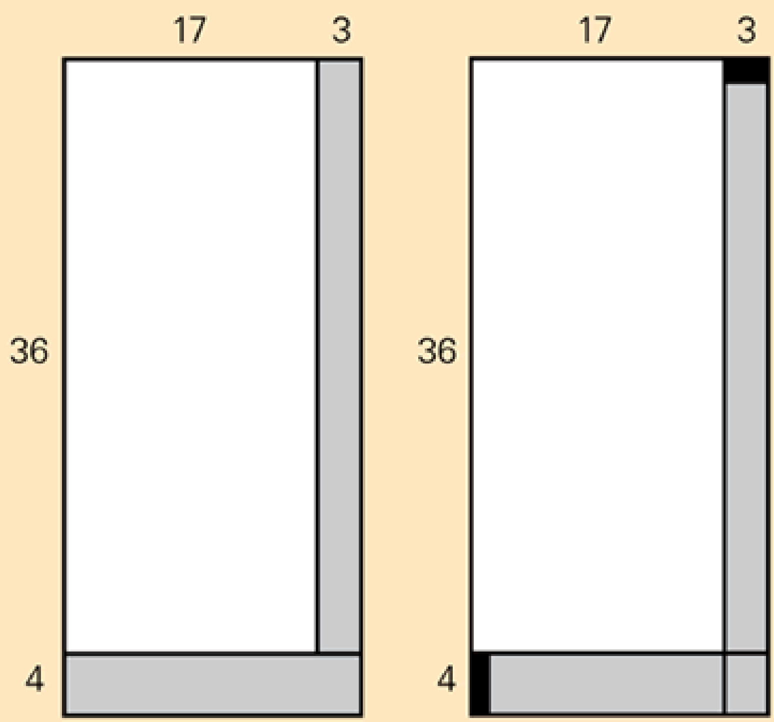
> 20, or $800. But on the day of the field trip, 4 students were absent. That means she needed to give back $20 to each of those children, 800 - (4 × 20). Then the teacher went to the museum with 36 children, but when they got there they realized that the entrance fee was $17 instead of $20. That meant that each of the remaining 36 children got $3 back. So now we have 800 - (4 × 20) - (36 × 3), which the teacher paid to the museum. And that's $612—$17 for each of 36 children, or 36 × 17.

Ming added, “If you think about what Thomas did, it's like he gave each of the 4 absent students only $1, and he gave only 1 other student $3.”

Chad offered his group's use of an array, or the area of a rectangle, to think through the problem (see fig. 1). He explained,

> The white part of the figures shows 36 × 17, and the gray regions show what gets added on when you change the problem to 40 × 20. In the picture on the right, you can see where Thomas went wrong. Instead of subtracting everything that got added on, he just took away what's shown in black.
>
> You can see Ming's story in the diagram on the left. The gray region at the bottom stands for the money that was returned to the 4 children who were absent. The gray region on the right is the money that was returned to the 36 children who went on the field trip. The white region is the money that was paid to the museum.

Figure 1. Chad's Diagram



Chad's group used an array, or the area of a rectangle, to explore different methods of finding the product of 36 x 17.

Annie pointed out that, when looking at Chad's diagram on the right, she can see more clearly why each of her initial answers was 12 off: "The first way I looked at it, I failed to subtract that little piece in the corner. The second way I looked at it, I subtracted that little piece twice."

Aisha offered a fourth way of viewing the problem:

> I wrote out the arithmetic and applied the distributive property: $(36 + 4) \times (17 + 3) = (36 \times 17) + (36 \times 3) + (4 \times 17) + (4 \times 3)$. So when Thomas multiplied 40×20, he needed to subtract those last three terms to get back to 36×17. When I was in high school, we called that procedure FOIL—you multiply the First terms, Outer terms, Inner terms, and Last terms. The thing is, I always did that because I was told that's the way to do it. But now that I can see it in the diagram, it really makes sense.

In this professional development session, participants offered four approaches to examine Thomas's strategy and figure out how to correct it. Note that, like Thomas, Annie chose to share her unresolved thinking. Looking together at what seems like it should be right, even though we know it's not, the teachers used several approaches to figure out where Annie's thinking went wrong. By sharing their different approaches, the teachers could compare approaches to see how one representation appeared in another.

The Professional Development Teachers Need

If teachers themselves were taught mathematics as discrete procedures and definitions to be memorized, how can schools prepare them to implement a more challenging, concept-based mathematics pedagogy? As a starting point, professional development needs to challenge teachers' conceptions of mathematics teaching and learning, opening them up to a process of reflection so that new insights can emerge.

Liz Sweeney's homework assignment provided just such an opportunity to the participants in the professional development seminar. Once the teachers had explored the mathematics in Thomas's error, they returned to their own questions about Sweeney's pedagogical approach. Among their comments were,

> Of course all students know that addition and multiplication are different, but they don't always think about that. Our exploration of Thomas's error really highlights how you have to think about multiplication differently.
>
> With these images, the distributive property isn't just a rule to memorize. You can see why it has to work.
>
> I bet Thomas felt proud to have presented something that got his classmates thinking so hard.

Such insights cannot be induced by a series of lectures or workshops on instructional strategies. Instead, professional development programs need to dig deeper, giving their participants opportunities to construct more powerful understandings of learning, teaching, and disciplinary substance.

A first step in helping teachers change their pedagogy is to place them in seminars where they can explore disciplinary content, develop new conceptions of mathematics, and gain a heightened sense of their own mathematical powers. As learners of mathematics, they experience a new kind of classroom. In these seminars, teachers reflect on their own learning processes and consider those features of the classroom that support or hinder them. Through such professional development, we can inspire teachers to envision and implement a new kind of mathematics pedagogy—one in which student understanding and collaborative thinking take center stage.

My “Aha!” Moment

Jeremy Kilpatrick, Regents Professor of Mathematics Education, University of Georgia, Athens. Winner of the National Council of Teachers of Mathematics Lifetime Achievement Award.

Although I did well in mathematics in high school, it was not until I went to Chaffey College, a two-year college then located in Ontario, California, and took calculus from Arthur E. Flum, that I discovered that learning mathematics could be simultaneously difficult and enjoyable, elegant and fascinating. The moment I realized all this came during the first week of class, when Mr. Flum's infectious enthusiasm for the subject we were about to work on together became apparent. Calculus was a new world for us, but under his guidance, we would succeed not only in learning it but in seeing its power and elegance. I ended up taking every mathematics course I could from Mr. Flum, and when I transferred as a junior to Cal Berkeley, mathematics was the obvious subject in which to major.

When I learned later that research on effective teachers has repeatedly shown that enthusiasm is one of their signature traits, I thought of Mr. Flum. In all that he did—coaching the tennis team, sponsoring the booster club, teaching mathematics—he had a flair for pushing you harder while helping you enjoy what you were doing. Successful mathematics teachers are enthusiastic about mathematics, and that enthusiasm is contagious.

Endnotes

1 This classroom episode can be seen in the video component of Schifter, D., Bastable, V., & Russell, S. J. (1999). *Building a system of tens*. Parsippany, NJ: Pearson.

2 The session described here is a composite of several seminar groups that were part of the *Developing Mathematical Ideas* professional development program.

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